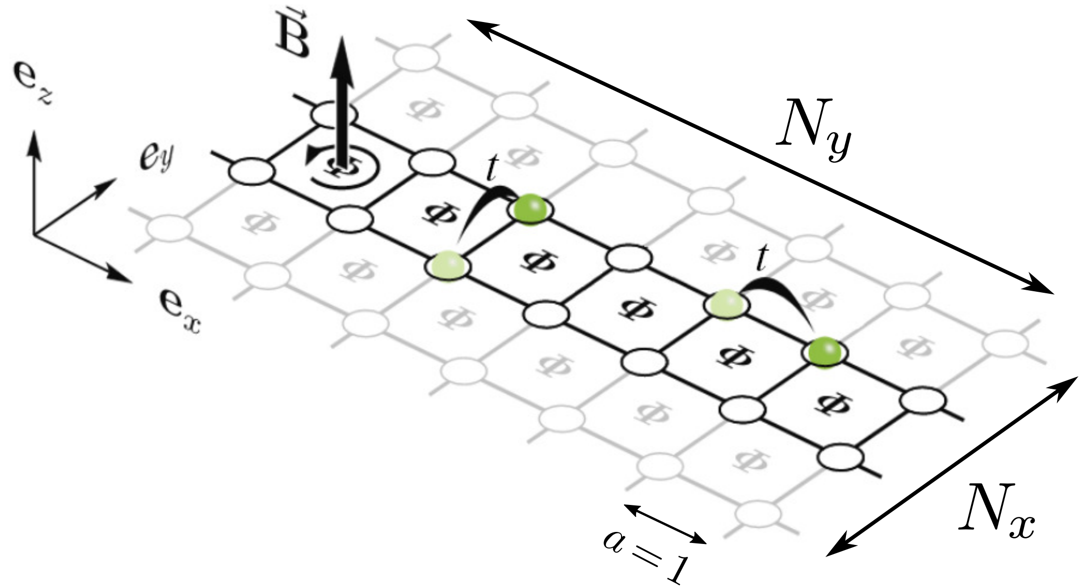
The Hofstadter model



One-orbital, spinless, nearest neighbor tight-binding model on a square lattice with p.b.c.

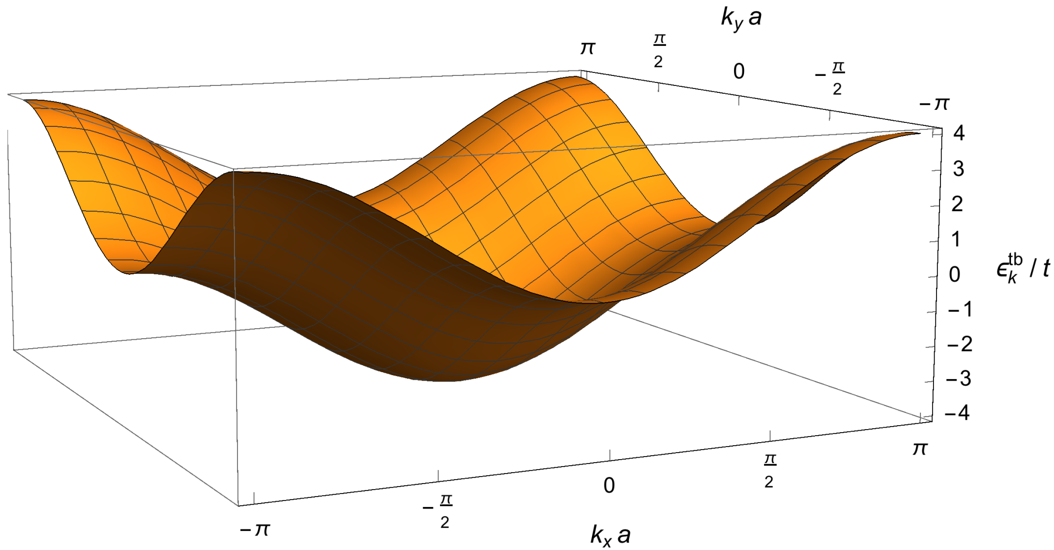
$$\boldsymbol{H} = -\sum_{\vec{r}} t\,\boldsymbol{c}^{\dagger}_{\vec{r}+\vec{e_x}}\,\boldsymbol{c}_{\vec{r}} + t\,\boldsymbol{c}^{\dagger}_{\vec{r}+\vec{e_y}}\,\boldsymbol{c}_{\vec{r}} + \mathrm{h.c.} + \begin{array}{c}\text{magnetic}\\\text{field term}\end{array}$$

Without a magnetic field

Bloch theorem ⇒ Diagonal hamiltonian in reciprocal space with

$$\tilde{\boldsymbol{c}}_{\vec{k}} := \frac{1}{\sqrt{N_x\,N_y}} \sum_{\vec{r}} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\,\boldsymbol{c}_{\vec{r}}$$

$\boldsymbol{H} = \sum_{\vec{k}} \epsilon(\vec{k})\,\tilde{\boldsymbol{c}}^{\dagger}_{\vec{k}}\,\tilde{\boldsymbol{c}}_{\vec{k}}$ with energy spectrum $\boxed{\epsilon(\vec{k}) = -2\,t\cos(k_x a) - 2\,t\cos(k_y a)}$



How to take into account $\vec{B}$? Semi-classical argument in path integral formalism

Transition amplitude : $\langle \vec{r}\,', t' | \vec{r}, t \rangle = \int_{\vec{r}(t)=\vec{r}}^{\vec{r}(t')=\vec{r}\,'} \mathcal{D}[\vec{r}(t)]\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S[\vec{r}(t)]}, \quad S[\vec{r}(t)] = \int_t^{t'} \mathrm{d}t\, \mathcal{L}(\vec{r}, \dot{\vec{r}})$

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charge (possibly < 0)

Including $\vec{B}$ in the lagrangian : $\mathcal{L} = \mathcal{L}^{(0)} + e\, \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \cdot \vec{A}$

$$
\begin{aligned}
S &= S^{(0)} + e \int_t^{t'} \mathrm{d}t\, \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \cdot \vec{A}(\vec{r}(t)) \\
&= S^{(0)} + e \int_{\vec{r} \to \vec{r}\,'} \mathrm{d}\vec{r} \cdot \vec{A}(\vec{r})
\end{aligned}
$$

Including the magnetic field : **Peierls substitution**

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Thus,

$$\langle \vec{r}', t' | \vec{r}, t \rangle \simeq \exp\left(\mathrm{i} \frac{e}{\hbar} \int_{\vec{r} \to \vec{r}'} \mathrm{d}\vec{r} \cdot \vec{A}(\vec{r}) \right) \cdot \int_{\vec{r}}^{\vec{r}'} \mathcal{D}[\vec{r}(t)]\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_0[\vec{r}(t)]}$$

Including the magnetic field : **Peierls substitution**

$$\langle \vec{r}\,', t' | \vec{r}, t \rangle \simeq \mathrm{e}^{\mathrm{i}\phi} \cdot \int_{\vec{r}}^{\vec{r}\,'} \mathcal{D}[\vec{r}(t)]\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_0[\vec{r}(t)]} \quad \text{with} \quad \phi = \frac{e}{\hbar} \int_{\vec{r} \to \vec{r}\,'} \mathrm{d}\vec{r} \cdot \vec{A}(\vec{r})$$

$$\Downarrow$$

$$\boxed{t\, \boldsymbol{c}^{\dagger}_{\vec{r}\,'} \boldsymbol{c}_{\vec{r}} \quad \mapsto \quad t\, \mathrm{e}^{\mathrm{i}\phi}\, \boldsymbol{c}^{\dagger}_{\vec{r}\,'} \boldsymbol{c}_{\vec{r}}}$$

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$$\langle \vec r', t' | \vec r, t\rangle \simeq \mathrm{e}^{\mathrm{i}\phi} \cdot \int_{\vec r}^{\vec r'} \mathcal{D}[\vec r(t)]\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_0[\vec r(t)]} \quad \text{with} \quad \phi = \frac{e}{\hbar}\int_{\vec r \to \vec r'} \mathrm{d}\vec r \cdot \vec A(\vec r)$$

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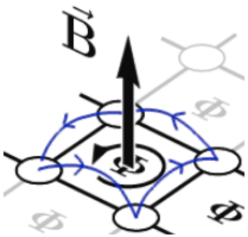

Hopping around a unit cell : accumulates a phase

$$\phi = \frac{e}{\hbar}\oint \overrightarrow{\mathrm{d}r} \cdot \vec A = \frac{e}{\hbar}\iint (\overbrace{\vec\nabla \times \vec A}^{\vec B}) \cdot \overrightarrow{\mathrm{d}^2\mathcal{S}} = \frac{e}{\hbar}\underbrace{\Phi}_{= B\, \mathcal{S}_{\text{u.cell}}}$$

$$\langle \vec{r}', t' | \vec{r}, t \rangle \simeq e^{i\phi} \cdot \int_{\vec{r}}^{\vec{r}'} \mathcal{D}[\vec{r}(t)]\, e^{\frac{i}{\hbar} S_0[\vec{r}(t)]} \quad \text{with} \quad \phi = \frac{e}{\hbar} \int_{\vec{r} \to \vec{r}'} d\vec{r} \cdot \vec{A}(\vec{r})$$

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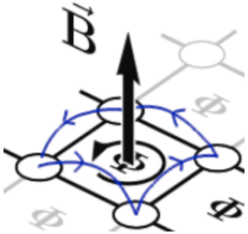

Hopping around a unit cell : accumulates a phase $\quad \vec{B}$

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$$= B\, \mathcal{S}_{\text{u.cell}}$$

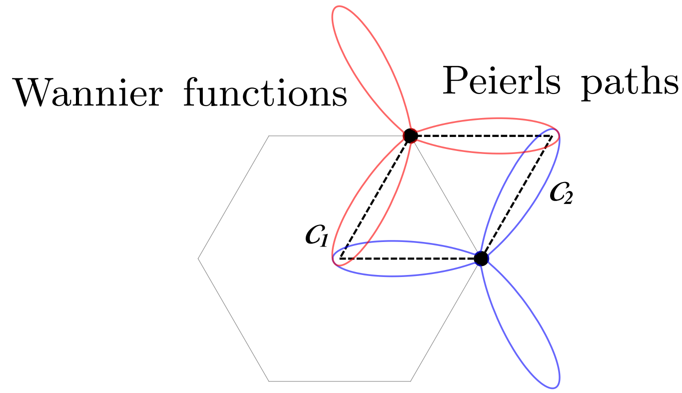
Hofstadter hamiltonian :

$$\boxed{\boldsymbol{H} = - \sum_{\langle r, \vec{r}' \rangle} t\, e^{i \frac{e}{\hbar} \int \vec{A} \cdot d\vec{r}}\, \boldsymbol{c}^\dagger_{\vec{r}'} \boldsymbol{c}_{\vec{r}} + \text{h.c.}}$$

- Full proof : go back to Wannier functions
- Actually “just” the consequence of enforcing $U(1)$ gauge symmetry in $\boldsymbol{H}$

Including the magnetic field : **Peierls substitution**

Warning, the Peierls path is not always the spatially shortest path (`arXiv:2006.13938`) :



Hofstadter hamiltonian in Landau gauge

Landau gauge $\vec{A} = [\ 0 \quad B\,x \quad 0\ ]$ and periodic boundary conditions :

- hopping $\vec{r} \rightarrow \vec{r} + \vec{e}_x$: $\quad \int \vec{A} \cdot \mathrm{d}\vec{r} = B \int_x^{x+a} x\, \vec{e_y} \cdot \vec{e_x}\, \mathrm{d}x = 0$
- hopping $\vec{r} \rightarrow \vec{r} + \vec{e}_y$: $\quad \int \vec{A} \cdot \mathrm{d}\vec{r} = B \int_y^{y+a} x\, \vec{e_y} \cdot \vec{e_y}\, \mathrm{d}y = B\,a\,x$ $\quad$ so that

$$\mathrm{e}^{\mathrm{i}\frac{e}{\hbar}\int \vec{A}\cdot \mathrm{d}\vec{r}}\, \boldsymbol{c}^\dagger_{\vec{r}+\vec{e}_y}\, \boldsymbol{c}_{\vec{r}} \;=\; \mathrm{e}^{\mathrm{i}\frac{e\,B\,a\,x}{\hbar}}\, \boldsymbol{c}^\dagger_{\vec{r}+\vec{e}_y}\, \boldsymbol{c}_{\vec{r}} \;=\; \mathrm{e}^{2\pi \mathrm{i} n_\phi \frac{x}{a}}\, \boldsymbol{c}^\dagger_{\vec{r}+\vec{e}_y}\, \boldsymbol{c}_{\vec{r}} \quad \text{with} \quad n_\phi = \frac{e\,B\,a^2}{h} = \frac{e}{h}\,\Phi_{\mathrm{u.cell.}}$$

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$$\boxed{\boldsymbol{H} \;=\; -\sum_{\vec{r}} t\, \boldsymbol{c}^\dagger_{\vec{r}+\vec{e}_x} \boldsymbol{c}_{\vec{r}} \;+\; t\,\mathrm{e}^{2\pi\mathrm{i} n_\phi x}\, \boldsymbol{c}^\dagger_{\vec{r}+\vec{e}_y} \boldsymbol{c}_{\vec{r}} \;+\; \mathrm{h.c.}}$$

with $a = 1$, i.e. $x \in \mathbb{N}$.

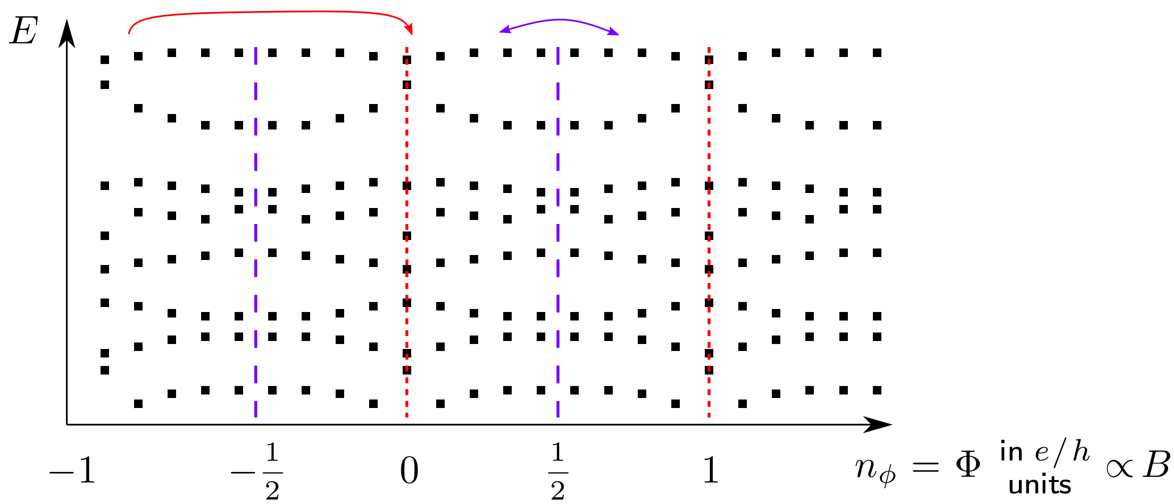
Some symmetries

$$\boldsymbol{H} = -\sum_{\vec{r}} t\, \boldsymbol{c}^{\dagger}_{\vec{r}+\vec{e}_x} \boldsymbol{c}_{\vec{r}} + t\, \mathrm{e}^{2\pi \mathrm{i} n_\phi x}\, \boldsymbol{c}^{\dagger}_{\vec{r}+\vec{e}_y} \boldsymbol{c}_{\vec{r}} + \text{h.c.}$$

We have $\mathrm{e}^{2\pi \mathrm{i} x} = 1$, so

- transformation $n_\phi \to n_\phi + 1$: $\boldsymbol{H}$ remains invariant $\Rightarrow$ we can restrict to $n_\phi \in [0, 1]$
- transformation $n_\phi \to 1 - n_\phi$: $\boldsymbol{H} = -\sum_{\vec{r}} t\, \boldsymbol{c}^{\dagger}_{\vec{r}+\vec{e}_x} \boldsymbol{c}_{\vec{r}} + t\, \mathrm{e}^{-2\pi \mathrm{i} n_\phi x}\, \boldsymbol{c}^{\dagger}_{\vec{r}+\vec{e}_y} \boldsymbol{c}_{\vec{r}} + \text{h.c.}$

  which does not change eigen-energies $\Rightarrow$ "reflection" symmetry



$n_\phi = p\,/\,q$

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Periodic boundary conditions along x impose that

$$\mathrm{e}^{2\pi \mathrm{i} n_\phi x}\big|_{x=N_x} = \mathrm{e}^{2\pi \mathrm{i} n_\phi x}\big|_{x=0}$$

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i.e.

$$n_\phi N_x \in \mathbb{N} \quad \Rightarrow \quad \boxed{n_\phi = \frac{n}{N_x},\ n \in \mathbb{N}}$$

Thus $n_\phi = \frac{p}{q} \in \mathbb{Q}$. We can always choose p, q coprime.

Flux is quantized in p.b.c. because we are on a closed surface ($\sim$ magnetic monopole).

Diagonalization of the Hofstadter hamiltonian

$$\boldsymbol{H} = -\sum_{x,y} t\,(\boldsymbol{c}^{\dagger}_{x+1,y}\,\boldsymbol{c}_{x,y} + \mathrm{e}^{2\pi \mathrm{i} n_{\phi} x}\,\boldsymbol{c}^{\dagger}_{x,y+1}\,\boldsymbol{c}_{x,y} + \mathrm{h.c.})$$

Still y-translation invariant. But the phase breaks the x-translation invariance…

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However, we can thus partially diagonalize the hamiltonian by going in Fourier space $y \to \tilde{k}_y$

Mixed annihilation operator :

$$\tilde{\boldsymbol{c}}_{x,\mathtt{k}_y} := \frac{1}{\sqrt{N_y}} \sum_y \mathrm{e}^{2\pi \mathrm{i} \frac{\mathtt{k}_y y}{N_y}}\,\boldsymbol{c}_{x,y} \quad \text{where} \quad \mathtt{k}_y \in [\![0, N_y]\!]$$

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We compute

$$\sum_{\mathtt{k}_y} \mathrm{e}^{2\pi \mathrm{i} \frac{\mathtt{k}_y}{N_y} n} \tilde{\boldsymbol{c}}^{\dagger}_{x,\mathtt{k}_y} \tilde{\boldsymbol{c}}_{x',\mathtt{k}_y} = \frac{1}{N_y} \sum_{y,y'} \sum_{\mathtt{k}_y} \mathrm{e}^{2\pi \mathrm{i} \frac{\mathtt{k}_y}{N_y}(-y+y'+n)} \tilde{\boldsymbol{c}}^{\dagger}_{x,y} \tilde{\boldsymbol{c}}_{x',y'} = \sum_{y} \tilde{\boldsymbol{c}}^{\dagger}_{x,y+n} \tilde{\boldsymbol{c}}_{x',y}$$

Diagonalization of the Hofstadter hamiltonian

$$\boldsymbol{H} = -\sum_{x,y} t\,(\boldsymbol{c}^{\dagger}_{x+1,y}\,\boldsymbol{c}_{x,y} + \mathrm{e}^{2\pi\mathrm{i}n_{\phi}x}\,\boldsymbol{c}^{\dagger}_{x,y+1}\,\boldsymbol{c}_{x,y} + \mathrm{h.c.})$$

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so

$$\boldsymbol{H} = -\sum_{x,\mathtt{k}_y} t\,(\tilde{\boldsymbol{c}}^{\dagger}_{x+1,\mathtt{k}_y}\,\tilde{\boldsymbol{c}}_{x,\mathtt{k}_y} + \mathrm{e}^{2\pi\mathrm{i}(n_{\phi}x+\mathtt{k}_y/N_y)}\,\tilde{\boldsymbol{c}}^{\dagger}_{x,\mathtt{k}_y}\,\tilde{\boldsymbol{c}}_{x,\mathtt{k}_y} + \mathrm{h.c.})$$

It is $\mathtt{k}_y$-diagonal !

Diagonalization of the Hofstadter hamiltonian

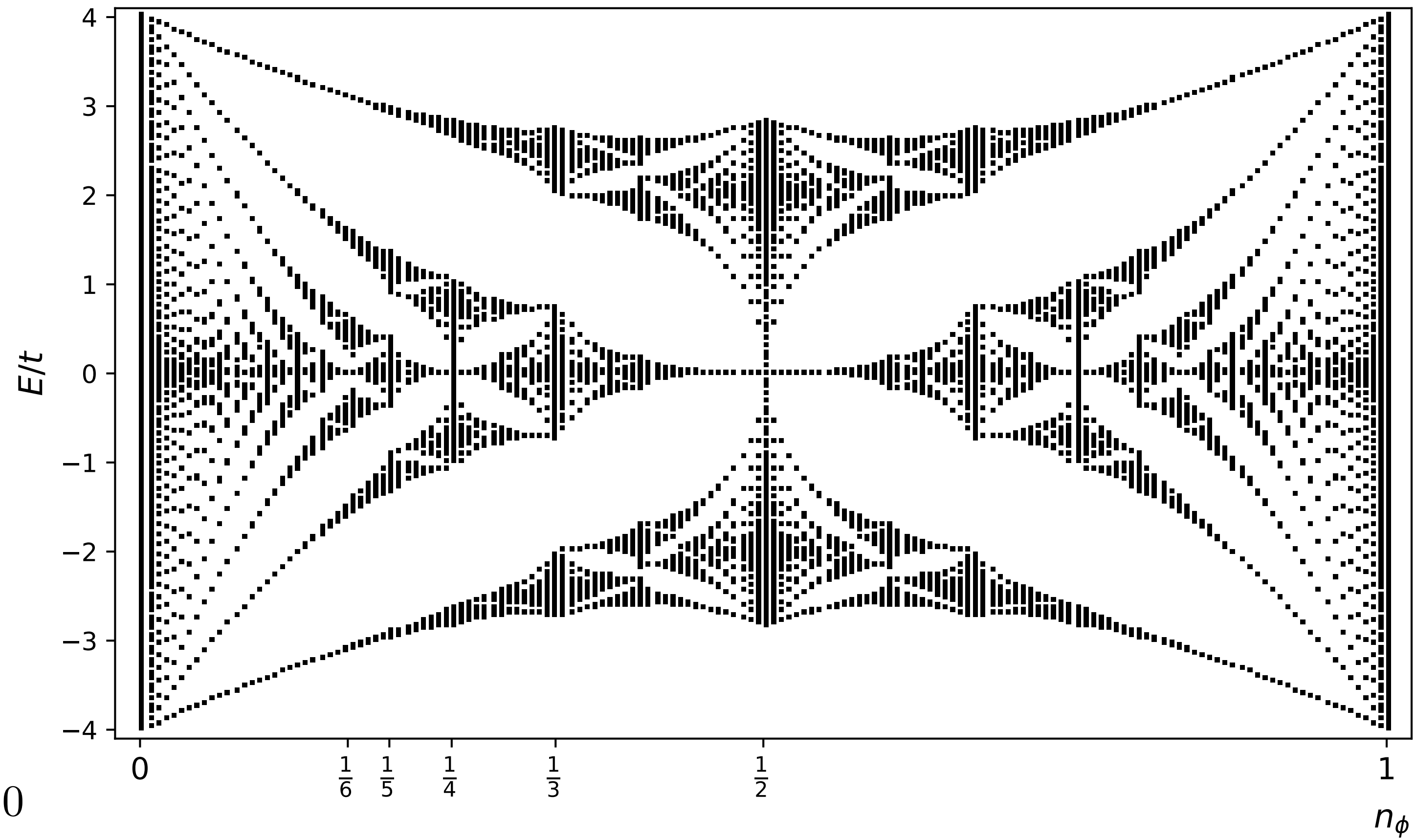
$$\begin{aligned}
\boldsymbol{H} &= -\sum_{x,\mathtt{k}_y} t\left(\tilde{\boldsymbol{c}}^{\dagger}_{x+1,\mathtt{k}_y}\tilde{\boldsymbol{c}}_{x,\mathtt{k}_y} + \mathrm{e}^{2\pi\mathrm{i}(n_\phi x+\mathtt{k}_y/N_y)}\,\tilde{\boldsymbol{c}}^{\dagger}_{x,\mathtt{k}_y}\tilde{\boldsymbol{c}}_{x,\mathtt{k}_y} + \mathrm{h.c.}\right) \\
&= -\sum_{x,\mathtt{k}_y} t\left(\tilde{\boldsymbol{c}}^{\dagger}_{x+1,\mathtt{k}_y}\tilde{\boldsymbol{c}}_{x,\mathtt{k}_y} + \tilde{\boldsymbol{c}}^{\dagger}_{x,\mathtt{k}_y}\tilde{\boldsymbol{c}}_{x+1,\mathtt{k}_y} + 2\,t\cos\left(2\pi\,n_\phi\,x + \mathtt{k}_y\frac{2\pi}{N_y}\right)\tilde{\boldsymbol{c}}^{\dagger}_{x,\mathtt{k}_y}\tilde{\boldsymbol{c}}_{x,\mathtt{k}_y}\right) \\
&= \sum_{\mathtt{k}_y}\boldsymbol{H}'(\mathtt{k}_y)
\end{aligned}$$

with

$$\boxed{\boldsymbol{H}'(\mathtt{k}_y) = -t\sum_{x}\tilde{\boldsymbol{c}}^{\dagger}_{x+1}\tilde{\boldsymbol{c}}_{x} + \tilde{\boldsymbol{c}}^{\dagger}_{x}\tilde{\boldsymbol{c}}_{x+1} + 2\cos\left(2\pi\,n_\phi\,x + \mathtt{k}_y\frac{2\pi}{N_y}\right)\tilde{\boldsymbol{c}}^{\dagger}_{x}\tilde{\boldsymbol{c}}_{x}}$$

Note : at first sight, $\mathtt{k}_y$ looks like an irrelevant phase (with p.b.c.), but it is not exactly.

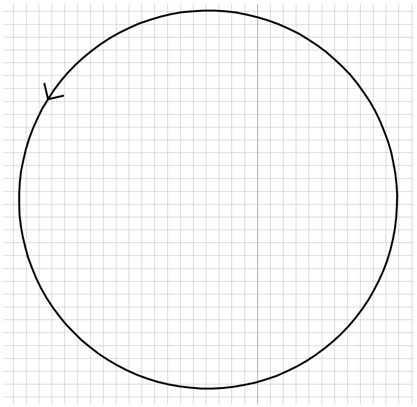
Hofstadter butterfly



$N_x = 160$
$N_y = 12$

Low field physics

At $B=0$, at the bottom of the band, the electron has an effective mass $m^*=\frac{\hbar^2}{t\,a^2}$.
At $B\neq 0$ such that $\ell_B=\sqrt{\hbar/e\,B}\ll a$:

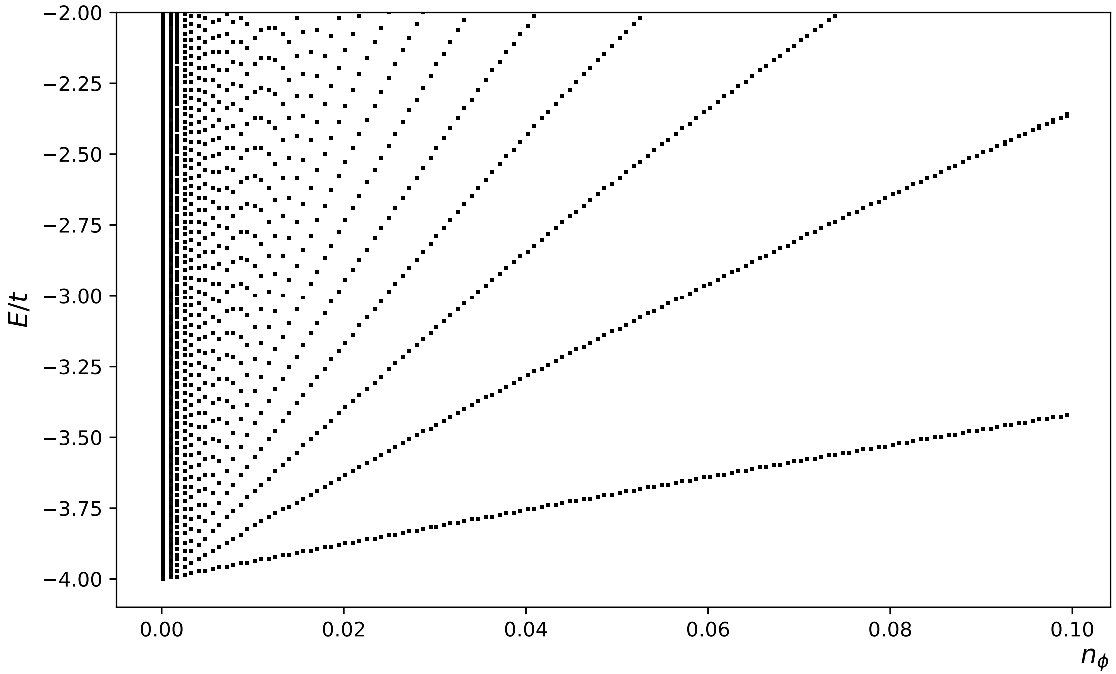


$\Rightarrow$ negligible lattice effects $\Rightarrow$ Landau levels

$$E_n\simeq\hbar\,\omega_c\left(n+\tfrac{1}{2}\right)\quad\text{with}\quad\omega_c=\frac{e\,B}{m^*}$$

Low field physics

Indeed, we observe a fan of Landau levels :



$$E_n \simeq \hbar\omega_c\left(n+\frac{1}{2}\right) \quad \text{with} \quad \omega_c \propto n_\phi$$

Justification :

$$\boldsymbol{H}'(\mathtt{k}_y) = -t\sum_x \tilde{\boldsymbol{c}}^\dagger_{x+a}\,\tilde{\boldsymbol{c}}_x + \tilde{\boldsymbol{c}}^\dagger_x\,\tilde{\boldsymbol{c}}_{x+a} + 2\cos\!\left(2\pi\, n_\phi\,\frac{x}{a} + \mathtt{k}_y\,\frac{2\pi\, a}{N_y}\right)\tilde{\boldsymbol{c}}^\dagger_x\,\tilde{\boldsymbol{c}}_x$$

Let $\boldsymbol{H}$ on a wavefunction $\psi(x, y)$:

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Let $\boldsymbol{H}$ on a wavefunction $\psi(x,y)$:
y-part is a Bloch wave, and $\boldsymbol{H}'(\mathtt{k}_y)$ acts on the x-part :

$$E\,\psi_{\mathtt{k}_y}(x) \;=\; \boldsymbol{H}'(\mathtt{k}_y)\,\psi_{\mathtt{k}_y}(x) \;=\; -t\left(\mathrm{e}^{-a\,\partial_x} + \mathrm{e}^{+a\,\partial_x} + 2\cos\Big(2\pi\, n_\phi\,\frac{x}{a} + \mathtt{k}_y\,\frac{2\pi\, a}{N_y}\Big)\right)\psi_{\mathtt{k}_y}(x)$$

(indeed, $\tilde{\boldsymbol{c}}_{x+a}^{\dagger}\,\tilde{\boldsymbol{c}}_x = (\boldsymbol{c}_{x,y}^{\dagger} + a\,\partial_x\,\boldsymbol{c}_{x,y}^{\dagger} + \cdots)\,\boldsymbol{c}_{x,y} = (\mathrm{e}^{a\,\partial_x}\,\boldsymbol{c}_{x,y}^{\dagger})\,\boldsymbol{c}_{x,y}$).

This is *Harper's equation*.

Low field physics

$$E\,\psi_{\mathtt{k}_y}(x) = \boldsymbol{H}'(\mathtt{k}_y)\,\psi_{\mathtt{k}_y}(x) = -t\left(\mathrm{e}^{-a\,\partial_x} + \mathrm{e}^{+a\,\partial_x} + 2\cos\left(2\pi\,n_\phi\,\frac{x}{a} + \mathtt{k}_y\,\frac{2\pi\,a}{N_y}\right)\right)\psi_{\mathtt{k}_y}(x)$$

Continuum limit $a \ll \ell_B \;\Leftrightarrow\; e\,B \ll \hbar/a^2$:

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Continuum limit $a \ll \ell_B \;\Leftrightarrow\; e\,B \ll \hbar/a^2$:

$\psi_{\mathtt{k}_y}(x)$ varies slowly (envelope function) and the $\cos$ can be expanded :

$$\begin{aligned}
E\,\psi_{\mathtt{k}_y}(x) &\simeq -t\left(2+a^2\,\partial_x^2+2-\left(\mathtt{k}_y\,\frac{2\pi\,a}{N_y}+2\pi\,n_\phi\,\frac{x}{a}\right)^2\right)\psi_{\mathtt{k}_y}(x)\\
\left[m^*=\tfrac{2\,\hbar^2}{t\,a^2},n_\phi=\tfrac{e\,B\,a^2}{h}\right]\qquad &= -\left(\mathrm{cst}-\frac{t\,a^2}{\hbar^2}\,\frac{(-\mathrm{i}\,\hbar\,\partial_x)^2}{2}-\frac{t\,a^2}{\hbar^2}\left(\frac{2\pi\,\hbar\,\mathtt{k}_y}{N_y}+e\,B\,x\right)^2\right)\psi_{\mathtt{k}_y}(x)\\
&= \frac{1}{2\,m^*}\left(\mathrm{cst}+\boldsymbol{p}_x^2+(\boldsymbol{p}_y^2+e\,B\,\boldsymbol{x})^2\right)\psi_{\mathtt{k}_y}(x)
\end{aligned}$$

Low field physics

$$E\,\psi_{\mathtt{k}_y}(x) = \boldsymbol{H}'(\mathtt{k}_y)\,\psi_{\mathtt{k}_y}(x) = -t\Big(\mathrm{e}^{-a\,\partial_x} + \mathrm{e}^{+a\,\partial_x} + 2\cos\Big(2\pi\,n_\phi\,\frac{x}{a} + \mathtt{k}_y\,\frac{2\pi\,a}{N_y}\Big)\Big)\,\psi_{\mathtt{k}_y}(x)$$

Continuum limit $a \ll \ell_B \;\Leftrightarrow\; e\,B \ll \hbar/a^2$:

$\psi_{\mathtt{k}_y}(x)$ varies slowly (envelope function) and the $\cos$ can be expanded :

$$\begin{aligned}
E\,\psi_{\mathtt{k}_y}(x) &\simeq -t\Big(2 + a^2\,\partial_x^2 + 2 - \Big(\mathtt{k}_y\,\frac{2\pi\,a}{N_y} + 2\pi\,n_\phi\,\frac{x}{a}\Big)^2\Big)\,\psi_{\mathtt{k}_y}(x)\\
\Big[m^* = \tfrac{2\,\hbar^2}{t\,a^2}, n_\phi = \tfrac{e\,B\,a^2}{h}\Big] \qquad &= -\Big(\mathrm{cst} - \frac{t\,a^2}{\hbar^2}\,\frac{(-\mathrm{i}\,\hbar\,\partial_x)^2}{2} - \frac{t\,a^2}{\hbar^2}\Big(\frac{2\pi\,\hbar\,\mathtt{k}_y}{N_y} + e\,B\,x\Big)^2\Big)\,\psi_{\mathtt{k}_y}(x)\\
&= \frac{1}{2\,m^*}\big(\mathrm{cst} + \boldsymbol{p}_x^2 + (\boldsymbol{p}_y^2 + e\,B\,\boldsymbol{x})^2\big)\,\psi_{\mathtt{k}_y}(x)
\end{aligned}$$

This is the Landau hamiltonian !

$$\boldsymbol{H}_L = \frac{1}{2\,m^*}\Big(\boldsymbol{p}_x^2 + (\boldsymbol{p}_y^2 + e\,B\,\boldsymbol{x})^2\Big)$$

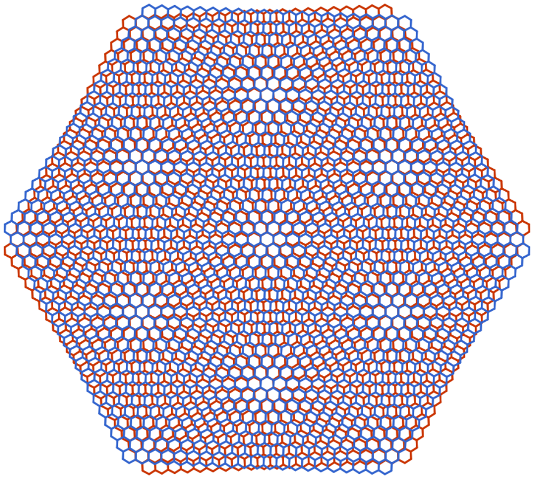
Are we bound to always be $n_\phi \ll 1$?

Cf. lecture :

$$\ell_B = \frac{26\,\mathrm{nm}}{\sqrt{B\,[T]}} \gg \quad a = \begin{cases} 0.142\,\mathrm{nm} & \text{for graphene} \\ 0.565\,\mathrm{nm} & \text{for GaAs} \end{cases}$$

$$\Leftrightarrow \qquad B \qquad \gg \qquad \begin{cases} 3 \cdot 10^4\,T & \text{for graphene} \\ 2 \cdot 10^3\,T & \text{for GaAs} \end{cases}$$

- In HbN substrate with Moire superlattice, high field ($\sim 40\ \mathrm{T}$) $\Rightarrow$ 0-energy Landau level $\Rightarrow$ massless Dirac fermions;
- While in graphene, 0 Landau level split into 2 non-0 energy levels.



Particle-hole symmetry

How to explain the $E \leftrightarrow -E$ symmetry ?

Consider the transformation $\Gamma : \boldsymbol{c}_{x,y} \mapsto (-1)^{x+y} \boldsymbol{c}_{x,y}$.

$$
\begin{aligned}
\boldsymbol{H} &= -t \sum_{x,y} \boldsymbol{c}^\dagger_{x+1,y} \boldsymbol{c}_{x,y} + \mathrm{e}^{2\pi \mathrm{i} n_\phi x} \boldsymbol{c}^\dagger_{x,y+1} \boldsymbol{c}_{x,y} + \text{h.c.} \\
&\mapsto -t \sum_{\vec{r}} \underbrace{(-1)^{2x+1+2y}}_{=-1} \boldsymbol{c}^\dagger_{x+1,y} \boldsymbol{c}_{x,y} + \underbrace{(-1)^{2x+2y+1}}_{=-1} \mathrm{e}^{2\pi \mathrm{i} n_\phi x} \boldsymbol{c}^\dagger_{x,y+1} \boldsymbol{c}_{x,y} + \text{h.c.} \\
&= -\boldsymbol{H} \quad \Rightarrow \quad \boxed{\Gamma\, \boldsymbol{H}\, \Gamma^{-1} = -\boldsymbol{H}}
\end{aligned}
$$

Consequence : if $\psi(x,y)$ is an eigenstate of energy E, then $(-1)^{x+y}\psi(x,y)$ is a *different* state of energy $-E$ $\Rightarrow$ *spectrum is symmetric around 0*.

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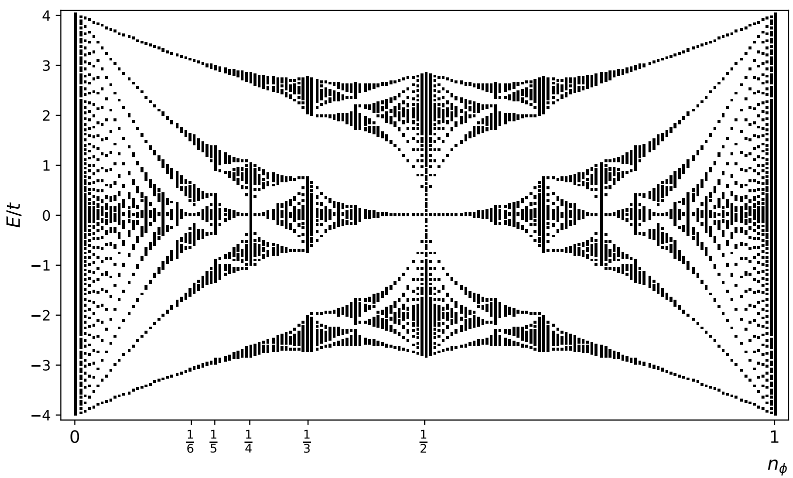
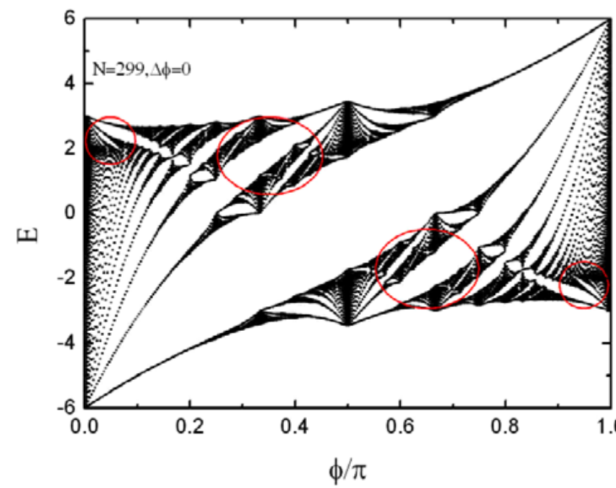
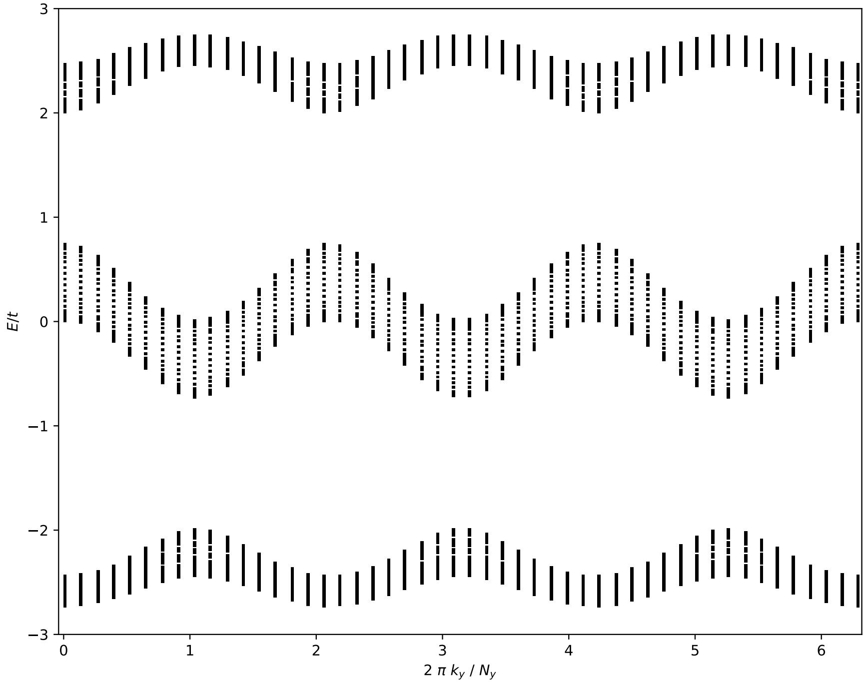
This is because a square lattice is bipartite. A triangular lattice would not have this symmetry :

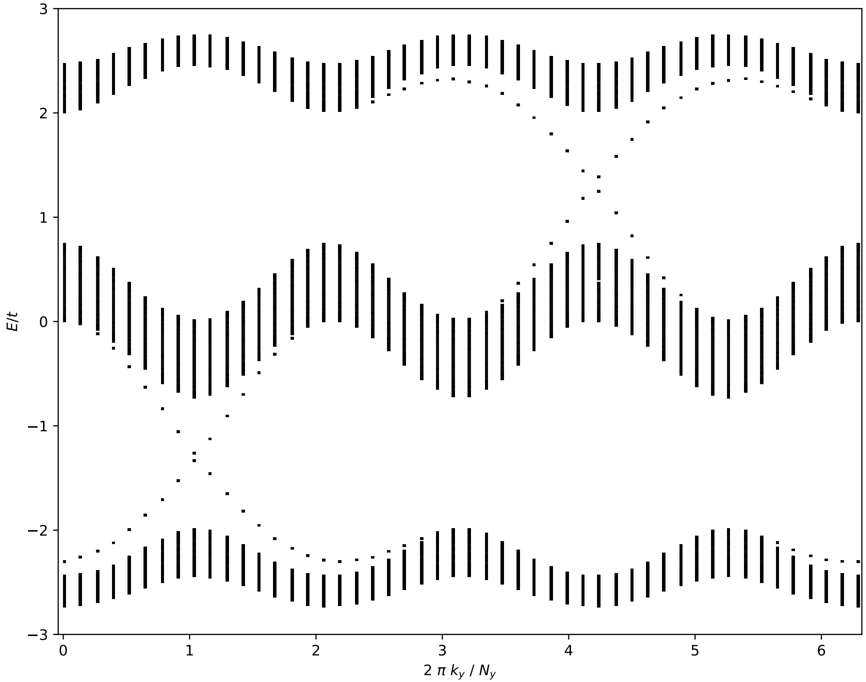




Open boundary conditions

$$N_x = 120; N_y = 60; n_\phi = \frac{1}{3}$$





Exploiting *"magnetic translation"* invariance

The x-translation invariance is broken... and a $N_x \times N_x$ hamiltonian is slow to diagonalize.

Exploiting *"magnetic translation"* invariance

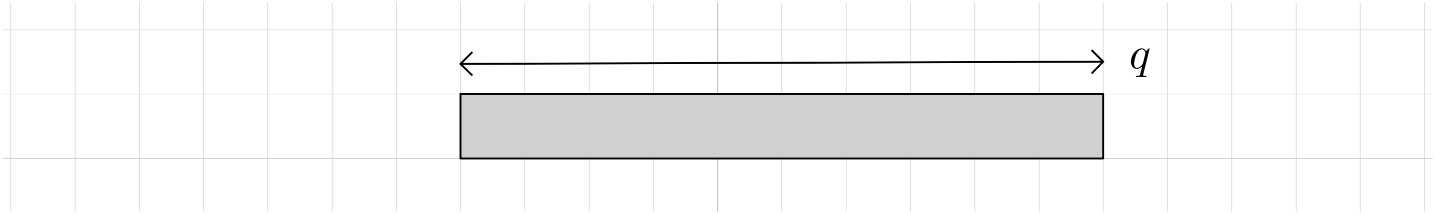
The x-translation invariance is broken… and a $N_x \times N_x$ hamiltonian is slow to diagonalize. But…

$$\boldsymbol{H} = -\sum_{x,y} t \left(\boldsymbol{c}^\dagger_{x+1,y} \boldsymbol{c}_{x,y} + \mathrm{e}^{2\pi \mathrm{i} n_\phi x} \boldsymbol{c}^\dagger_{x,y+1} \boldsymbol{c}_{x,y} + \mathrm{h.c.} \right)$$

is still invariant when we translate along x by $\boldsymbol{q}$ **sites** $(n_\phi = {}^p\!/_q)$:

$$\mathrm{e}^{2\pi \mathrm{i} n_\phi x} \boldsymbol{c}^\dagger_{x,y+1} \boldsymbol{c}_{x,y} \quad \mapsto \quad \underbrace{\mathrm{e}^{2\pi \mathrm{i} n_\phi (x+q)}}_{\substack{= \mathrm{e}^{2\pi \mathrm{i} p \frac{x+q}{q}} \\ = \mathrm{e}^{2\pi \mathrm{i} p \frac{x}{q}} \quad = \mathrm{e}^{2\pi \mathrm{i} n_\phi x}}} \boldsymbol{c}^\dagger_{x+q,y+1} \boldsymbol{c}_{x+q,y}$$

+ p.b.c.



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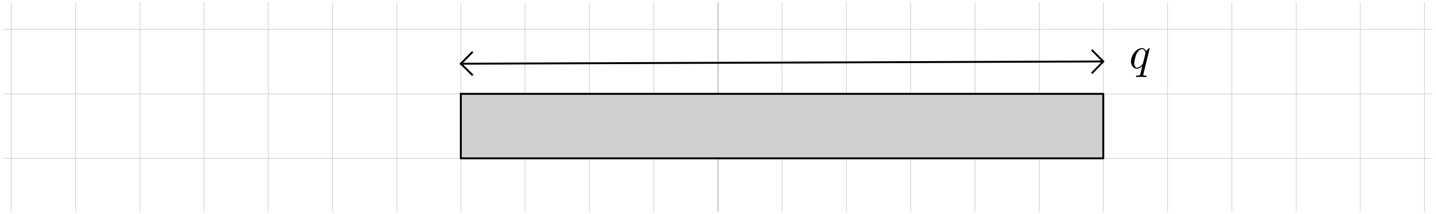

Bloch hamiltonian with unit cell $q \times 1$ (with p flux quanta) $\Rightarrow$ BZ q-folded

[play with the code, looking at the spectrum for various small fields]

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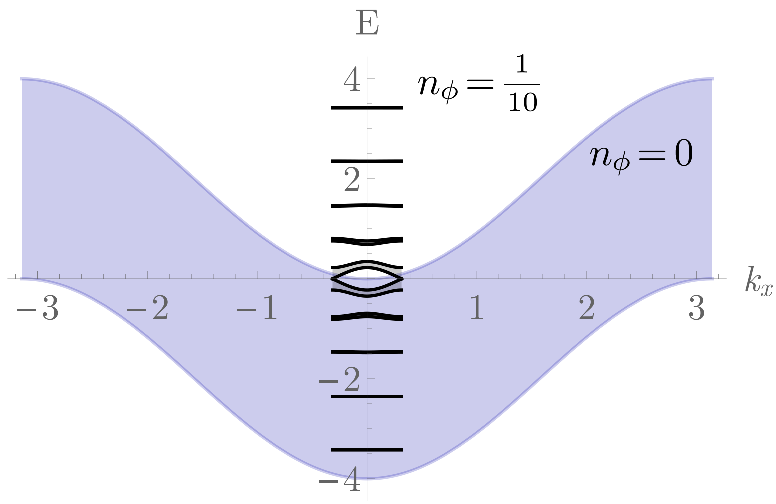

Exploiting *"magnetic translation"* invariance

$$\boldsymbol{H}'(\mathtt{k}_y) \;=\; -t\sum_x \tilde{\boldsymbol{c}}^\dagger_{x+1}\,\tilde{\boldsymbol{c}}_x \;+\; \tilde{\boldsymbol{c}}^\dagger_x\,\tilde{\boldsymbol{c}}_{x+1} + 2\cos(2\pi\,n_\phi\,x + k_y)\,\tilde{\boldsymbol{c}}^\dagger_x\,\tilde{\boldsymbol{c}}_x \qquad \left(k_y = \mathtt{k}_y\,\frac{2\pi}{N_y}\right)$$

$$\sum_x \cos(2\pi\,n_\phi\,x)\ldots = \sum_j \cos(2\pi\,n_\phi\,j)\sum_n \ldots \qquad j\in[\![1,q]\!] \qquad x = n\,q + j$$

Introduce

$$\tilde{\tilde{\boldsymbol{c}}}_{j,k_x,k_y} := \frac{1}{\sqrt{q}}\sum_n \mathrm{e}^{2\pi\mathrm{i}k_x n}\,\tilde{\boldsymbol{c}}_{nq+j,k_y}, \qquad \tilde{\tilde{\boldsymbol{c}}}_{q,k_x,k_y} \equiv \tilde{\tilde{\boldsymbol{c}}}_{0,k_x,k_y}$$

$$\boldsymbol{H}'(k_y) \;=\; -t\sum_{k_x} \boldsymbol{H}''(k_x,k_y)$$

with

$$\boldsymbol{H}'' = \sum_j \mathrm{e}^{-2\pi\mathrm{i}k_x}\,\tilde{\tilde{\boldsymbol{c}}}^\dagger_{j+1,k_x}\,\tilde{\tilde{\boldsymbol{c}}}_{j,k_x} + \mathrm{e}^{2\pi\mathrm{i}k_x}\,\tilde{\tilde{\boldsymbol{c}}}^\dagger_{j,k_x}\,\tilde{\tilde{\boldsymbol{c}}}_{j+1,k_x} + 2\cos(2\pi\,n_\phi\,j + k_y)\,\tilde{\tilde{\boldsymbol{c}}}^\dagger_{j,k_x}\,\tilde{\tilde{\boldsymbol{c}}}_{j,k_x}$$